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## Second Derivative Test:

Spse that the second partial derivatives are continuous on a disc w/ center $(a,b)$, and suppose $f_x(a,b)=0$ and $f_y(a,b)=0$.

Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local min.

b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local max

c) If $D<0$, then $f(a,b)$ is not a local max or min (saddle point) and graph of $f$ crosses the tangent plane at $(a,b)$.

RmK $D=0$ means test is inconclusive.

Ex Find the local max, min and saddle points of $f(x,y) = x^4+y^4-4xy+1$

Step 1 $f_x = 4x^3-4y$, $f_y = 4y^3-4x$

Setting these partial derivatives equal to 0, we get

$x^3-y=0$ and $y^3-x=0$

$y=x^3$

$\Rightarrow x^9-x=0 \Rightarrow x(x^8-1)=0 \Rightarrow x(x-1)(x+1)(x^2+1)(x^4+1)=0$

$\Rightarrow x=0,1,-1 \Rightarrow (0,0),(1,1),(-1,-1)$

$f_{xx}=12x^2$, $f_{xy}=-4$, $f_{yy}=12y^2$

$D(x,y) = f_{xx}f_{yy}-[f_{xy}]^2 = 144x^2y^2-16$

At $(0,0)$, $D(0,0)=-16<0$, saddle point at $(0,0,0)$

At $(1,1)$, $D(1,1)=128>0$, $f_{xx}(1,1)=12>0$, rel min $(1,1,-1)$

At $(-1,-1)$, $D(-1,-1)=128>0$, $f_{xx}(-1,-1)=12>0$, local min $(-1,-1,-1)$

## Absolute maximum and minimum values

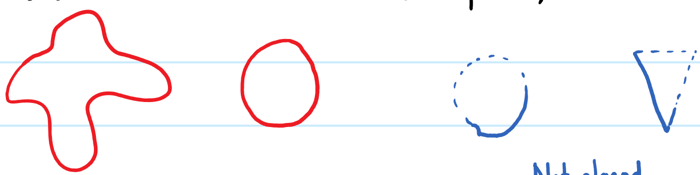
For a function $f$ of one variable

**Extreme Value Thm** : If $f$ is continuous on a closed interval $[a,b]$, then $f$ has an absolute minimum value and an absolute min value.

**Closed Interval Method** : Evaluate $f$ at critical numbers but also at endpoints $a$ and $b$.

Similar situation for function of two variables :

Just as a closed interval contains it's endpoints, a closed set contains all it's boundary points.



closed sets

Not closed.

$B_\varepsilon(\vec{x}) = \{ \vec{y} \in \mathbb{R}^n \mid |\vec{x}-\vec{y}| < \varepsilon \}$ is not closed (open ball : Every point is an interior point)

$\overline{B_\varepsilon}(\vec{x}) = \{ \vec{y} \in \mathbb{R}^n \mid |\vec{x}-\vec{y}| \leq \varepsilon \}$ is closed. (Closed ball : contains the boundary).

- A bounded set in $\mathbb{R}^n$ is one that can be contained within some ball.

## Extreme Value Thm for functions of Two variables

If $f$ is continuous on a closed bounded set $D$ in $\mathbb{R}^2$, then $f$ attains an absolute max. value $f(x_1,y_1)$ and an absolute min value $f(x_2,y_2)$ at some points $(x_1,y_1)$ and $(x_2,y_2)$ in $D$.

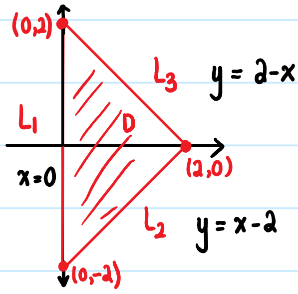
- **Remark** The extreme values can occur at either a critical point of $f$ or the bdry pt of $D$

To find absolute max and min values of a continuous function $f$ on a closed and bounded set $D$:

1) Find the values of $f$ at the critical points of $f$ in $D$.
2) Find the extreme values of $f$ on the boundary of $D$.
3) Find the largest and smallest values from Step 1 and 2.

Ex Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x$ on the closed triangular region with vertices $(2,0), (0,2)$ and $(0,-2)$.

Soln




So $D$ is closed and bounded, and $f$ is a polynomial and therefore it is continuous on $D$.

1) $f_x = 2x - 2$, $f_y = 2y$

Setting $f_x = f_y = 0 \Rightarrow x = 1, y = 0$.
Therefore,
$(1,0)$ is critical point (which is inside $D$), and $f(1,0) = -1$.

2) We look at the values of $f$ on the boundary of $D$, which consists of 3 line segments, $L_1, L_2, L_3$.

On $L_1$, we have $x = 0$ and
$f(0,y) = y^2$ and $-2 \le y \le 2$. (The problem reduces to EVT of a single variable function).

Call it $h(y) = y^2$ and $-2 \le y \le 2$.
Then $h'(y) = 2y = 0 \Rightarrow y = 0$
$h(0) = 0$, $h(2) = 4$, $h(-2) = 4$.

So along $L_1$, min at $(0,0)$ and $f(0,0)=0$

max at $(0,\pm 2)$ and $f(0,\pm 2) = 4$.

<u>Along $L_2$</u> : $y = x-2$ for $0 \le x \le 2$

and $f(x,x-2) = \underset{h(x)}{\underbrace{2x^2-6x+4}}$ , $0 \le x \le 2$

$h'(x) = 4x-6 = 0 \Rightarrow x = \frac{3}{2}$

$h\left(\frac{3}{2}\right) = -\frac{1}{2}$ , $h(0) = 4$ , $h(2) = 0$

• <u>On $L_2$</u> $f$ attains min at $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and $f\left(\frac{3}{2}, -\frac{1}{2}\right) = 1$

max at $(0,-2)$ and $f(0,-2) = 4$.

<u>Along $L_3$</u> : $y = 2-x$, $0 \le x \le 2$ and

$f(x, 2-x) = 2x^2-6x+4$ which is same as $L_2$.

• <u>On $L_2$</u> $f$ attains min at $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and $f\left(\frac{3}{2}, -\frac{1}{2}\right) = 1$

max at $(0,-2)$ and $f(0,-2) = 4$.

The abs max on $D$ is $f(0,\pm 2) = 4$

abs min on $D$ is $f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}$

<u>Maximizing Directional Derivative</u>

<u>Thm</u> Suppose $f$ is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same dirn as the gradient vector $\nabla f(\vec{x})$.

## Level surfaces

Recall For a function of two variables $f(x,y)$, the level curves of $f$ are the curves w/ equations $f(x,y) = k$, for constant $k$ in the range of $f$.

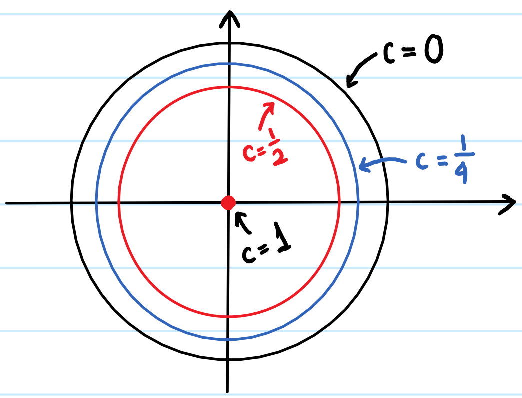
Ex Let $f(x,y) = \sqrt{1-x^2-y^2}$. Describe the level curves of $f$.

Solution Let $c \in \mathbb{R}$ and to describe the $c$ valued level curves we consider $c = \sqrt{1-x^2-y^2}$. Note that $c > 0$.

$\Rightarrow c^2 = 1-x^2-y^2$

$\Rightarrow x^2+y^2 = 1-c^2$. Therefore for $c$, we see that the level curves are circle centered at $(0,0)$ w/ radius $\sqrt{1-c^2}$.

The cases $c = 0, \frac{1}{2}, \frac{1}{4}, 1$




Then for a function of 3 variable $F(x,y,z)$, a level surface $S$ is a surface with the equation $F(x,y,z) = k$.

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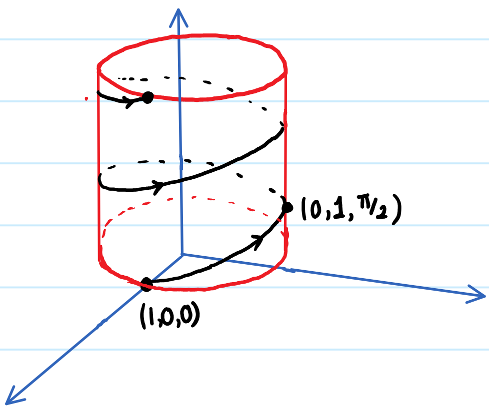
Recall A curve $C$ is described by a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Ex $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$

The parametric equations for this curve are :

$x = \cos t$ , $y = \sin t$ , $z = t$




Let $P_0(x_0, y_0, z_0)$ be a point on a level surface $S$ w/ equation $F(x,y,z) = k$.

Let $C$ be a curve that lies on the surface $S$ and passes through point $P$.

Let $P_0 = \vec{r}(t_0) = (x_0, y_0, z_0)$.

• Since $C$ lies on the surface $S$, $F(\vec{r}(t)) = k$ i.e. $F(x(t), y(t), z(t)) = k$.

If $x(t), y(t)$ and $z(t)$ are differentiable and if $F$ is diff, then $\frac{dF}{dt} = 0$ i.e

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial t} = 0$$

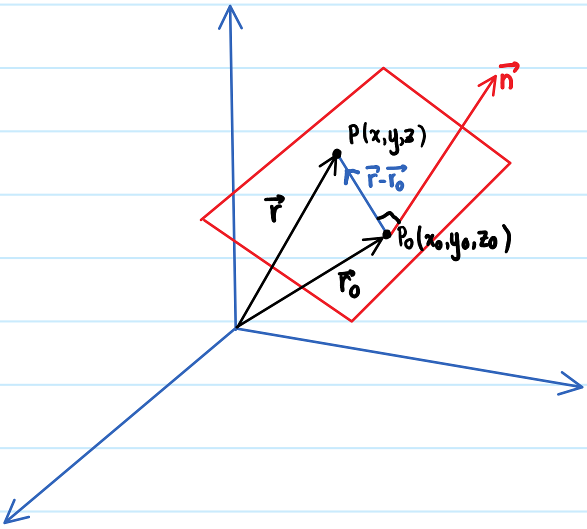
$\nabla F \cdot \vec{r}\,'(t) = 0$. For when $t = t_0$, $\nabla F(x_0, y_0, z_0)\,\vec{r}\,'(t_0) = 0$.

In words, the gradient vector at $P$, $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{r}\,'(t_0)$

to any curve $C$ on $S$ that passes through $P$.

If $\nabla F(x_0, y_0, z_0) \neq 0$, then it is natural to define the tangent plane to the level surface at $P(x_0, y_0, z_0)$ as the plane that passes through $P$ and has normal vector $\nabla F(x_0, y_0, z_0)$.

<u>Recall</u> A plane in space is determined by a point $P(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is <u>orthogonal</u> to the plane ($\vec{n}$ is called the normal vector)




Pick an arbitrary point $P(x,y,z)$ on the plane and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of $P_0$ and $P$. Then the vector $\overrightarrow{P_0P}$ can be expressed as $\vec{r}-\vec{r_0}$ (using the triangle law).

The normal vector $\vec{n}$ is orthogonal to every vector in the given plane, and in this case to $\vec{r}-\vec{r_0}$ and so we have

$$\boxed{\vec{n}\cdot(\vec{r}-\vec{r_0}) = 0}$$

If $\vec{n} = \langle a,b,c \rangle$, then we have

$$\langle a,b,c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

or, $$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

<u>Note</u> Two planes are parallel if their normal vectors are parallel.

Then going back to original question, we can write the equation of the tangent plane as :

$$F_x(x_0,y_0,z_0)\cdot(x-x_0) + F_y(x_0,y_0,z_0)\cdot(y-y_0) + F_y(x_0,y_0,z_0)\cdot(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

## Gradient vector (Pg 942).

Consider a function $f$ of 3 variables and a point $P(x_0,y_0,z_0)$ in it's domain.

1) $\nabla f(x_0,y_0,z_0)$ gives the direction of fastest increase of $f$.
2) $\nabla f(x_0,y_0,z_0)$ is orthogonal to the level surface $S$ of $f$ through $P$.

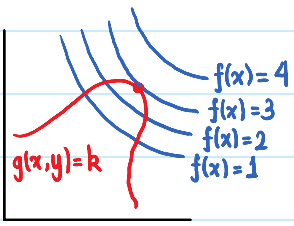
- So as we move away from $P$ along the level surface $S$, $f$ doesn't change. Therefore it seems reasonable that if we move in the perpendicular direction, we get maximum increase.

## Lagrange Multipliers

Let's say we want to find extreme values of $f(x,y)$ subject to a constraint of the form $g(x,y) = k$.

Said differently, we want to find extreme values of $f(x,y)$, when the point is restricted to level curves $g(x,y) = k$.

In other words, we want to find the largest value of $c$ such that the level curve $f(x,y) = c$ intersects $g(x,y) = k$.




- Idea

In the example, it occurs when they just touch each other, i.e. when they have a common tangent line. (Otherwise increase $c$)

- This means that their normal lines are identical at point $(x_0,y_0)$. $\Rightarrow$ Gradient vectors are parallel i.e

$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0) \text{ for some } \lambda \in \mathbb{R}.$$

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Suppose $f(x,y,z)$ has an extreme val. at a point $P(x_0,y_0,z_0)$ on the level surface $S$ w/ equation $g(x,y,z)=k$.

Let $C$ be a curve on $S$ given by $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$ and $\vec{r}(t_0)=\langle x_0,y_0,z_0\rangle$.

Then, $h(t)=f(\vec{r}(t))=f(x(t),y(t),z(t))$ represents the value that $f$ takes on the curve $C$.

$f$ has an extreme value at $P_0 \Rightarrow h$ has an extreme value at $t_0 \Rightarrow h'(t_0)=0 \Rightarrow$

$\Rightarrow \nabla f_x(x_0,y_0,z_0).\vec{r}'(t_0)=0$ (Just the Chain rule, Case 1).

i.e. $\nabla f(x_0,y_0,z_0)$ is orthogonal to $\vec{r}'(t_0)$, to every such curve $C$.

But we already Know from construction that $\nabla g(x_0,y_0,z_0)$ is orthogonal to $\vec{r}'(t_0)$ for every such curve.

Therefore, $\nabla f(x_0,y_0,z_0)$ is parallel to $\nabla g(x_0,y_0,z_0)$.

In particular, if $\nabla g(x_0,y_0,z_0)\neq 0$, there there is a number $\lambda$ such that:

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0).$$

$\lambda \equiv$ Lagrange multiplier.

<u>Method of Lagrange multipliers</u>

To find the maximum and minimum values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$ (assuming that these extreme values exist and $\nabla g\neq 0$ on the surface $g(x,y,z)=k$]

a) Find all values of $x,y,z$ and $\lambda$ such that

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$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = k.$$

b) Evaluate $f$ at all points $(x,y,z)$ from Step a).
The largest of these values is the maximum value of $f$; the smallest the minimum value.

$$\nabla f = \lambda \nabla g \Rightarrow \langle f_x, f_y, f_z \rangle = \lambda \cdot \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

$\Rightarrow f_x = \lambda g_x,\ f_y = \lambda g_y,\ f_z = \lambda g_z$ Together w/ $g(x,y,z) = k$.

For a function of 2 variables, Lagrange multipliers is similar.

To find extreme values of $f(x,y)$ subject to constraint $g(x,y) = k$, look for values $x, y$ and $\lambda$ s.t.

$$\nabla f(x,y) = \lambda g(x,y) \text{ and } g(x,y) = k.$$

<u>RmK</u> Not necessary to find explicit values of $\lambda$.

<u>Two constraints</u>

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$$
$$g(x,y,z) = k$$
$$h(x,y,z) = c$$

# 11

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**Question 1.** Your automobile assembly plant has a **Cobb-Douglas production function** given by

$$q = 100x^{0.3}y^{0.7}$$

where $q$ is the number of automobiles it produces per year, $x$ is the number of employees, and $y$ is the monthly assemblyline budget (in thousands of dollars).

Annual operating costs amount to an average of \$60 thousand per employee plus the operating budget of \$$12y$ thousand. Your annual budget is \$1, 200, 000. How many employees should you hire and what should your assembly-line budget be to maximize productivity? What is the productivity at these levels?

Answer

The objective is to maximize the productivity:

$$q = 100x^{0.3}y^{0.7} \text{ subject to } 60x + 12y = 1200, x \geq 0, y \geq 0$$

So we want to use the method of Lagrange Multipliers.

So, $g(x, y) = 60x + 12y - 1200$ and $f(x, y) = 100x^{0.3}y^{0.7}$.

Therefore the system of equations we want to solve is:

$$\begin{aligned} f_x = \lambda g_x : \quad & 30x^{-0.7}y^{0.7} = 60\lambda \\ f_y = \lambda g_y \quad & 70x^{0.3}y^{-0.3} = 12\lambda \\ g = 0 : \quad & 60x + 12y - 1200 = 0 \end{aligned}$$

Then we can simply the first and second questions are rewrite them as

$$30\left(\frac{y}{x}\right)^{0.7} = 60\lambda \quad 70\left(\frac{x}{y}\right)^{0.3} = 12\lambda$$

Now to solve the system of equations, we divide the first equation by the second to obtain:

$$\begin{aligned} & \frac{3}{7}\left(\frac{y}{x}\right)^{0.7}\left(\frac{y}{x}\right)^{0.3} = 5 \\ \Longrightarrow & \frac{3}{7}\left(\frac{y}{x}\right) = 5 \\ \Longrightarrow & y = \frac{35}{3}x. \end{aligned}$$

Finally substituting this into the constraint equation gives

$$60x + 12\left(\frac{35}{3}\right)x = 1200 \implies 200x = 1200 \implies x = 6 \text{ employees}$$

$$y = \frac{35}{3}x = \$70 \text{ thousand}$$

$$q = 100(6)^{0.3}(70)^{0.7} = 3,350 \text{ automobiles per year}$$

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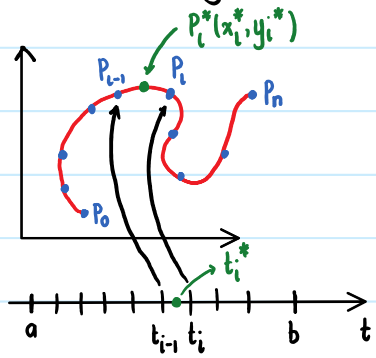
## 16.2 Line Integrals

- Define an integral similar to a single integral except we integrate over a curve $C$, rather than over an interval $[a,b]$.

- Start with a plane curve $C$ given by the parametric equations $\vec{r}(t) = \langle x(t), y(t) \rangle$.
  $x = x(t)$, $y = y(t)$, $a \le t \le b$.

Assume that $C$ is smooth i.e $r'(t) = \langle x'(t), y'(t) \rangle$ is continuous and $\vec{r}'(t) \neq 0$ on $I$.

If we divide the parameter interval $[a,b]$ into $n$ subintervals $[t_{i-1}, t_i]$ of equal width and let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P(x_i, y_i)$ divide $C$ into $n$ subarcs w/ lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$.




Choose any point $P_i^*(x_i^*, y_i^*)$ in the $i^{th}$ subarc (This corresponds to a point $t_i^*$ in $[t_{i-1}, t_i]$).

If $f$ is a function of two variables whose domain includes the curve $C$, evaluate $f$ at the point $(x_i^*, y_i^*)$, multiply by the length $\Delta s_i$ of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

<u>Def</u> If $f$ is defined on a smooth curve $C$ given by $x = x(t)$, $y = y(t)$, $a \le t \le b$, then the line integral of $f$ along $C$ is

$$\int_C f(x,y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

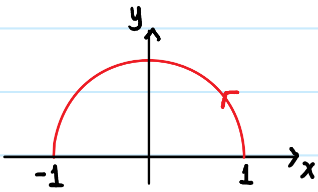
the length of $C$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Then we can show that,

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

<u>Ex</u> Evaluate $\int_C (2+x^2y)\,ds$ , where $C$ is the upper half of the unit circle $x^2+y^2=1$.




First we need to parametrize $C$.

$x = \cos t$ , $y = \sin t$ where $0 \le t \le \pi$.

Then, $\int_C (2+x^2y)\,ds = \int_0^\pi (2+\cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^\pi (2+\cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^\pi (2+\cos^2 t \sin t)\, dt = \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi$$

$$= 2\pi + \frac{2}{3}.$$